International macroeconomics

Additional material

Nikolas A. Müller-Plantenberg[1]

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[1] E-mail: nikolas@mullerpl.net. Dirección: Departamento de Análisis Económico - Teoría Económica e Historia Económica, Universidad Autónoma de Madrid, 28049 Madrid, España.

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1.1 Preliminary concepts

1.1.1 Logarithmic approximations

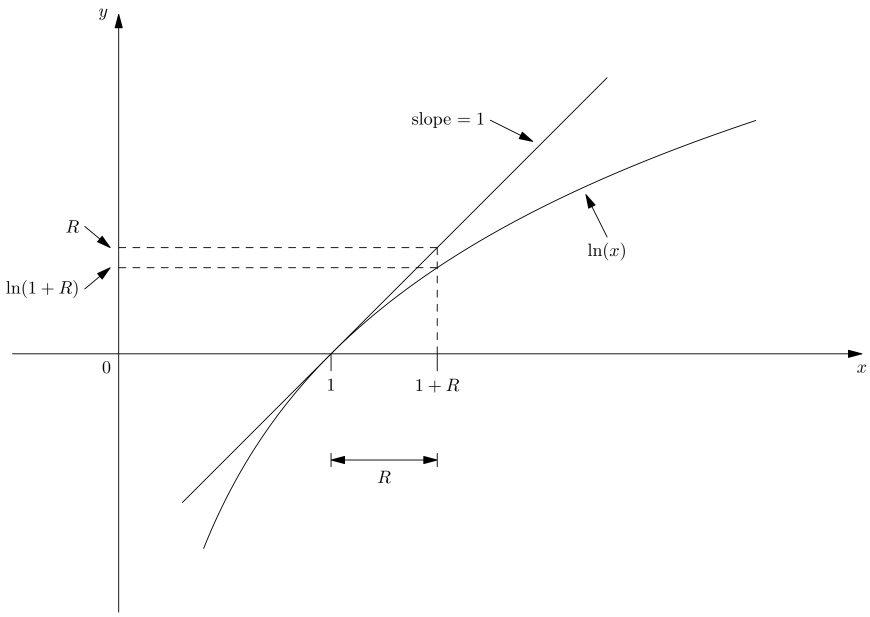

Figure 1.1: Log approximation.

1.1.2 The natural logarithm

The number e is defined as follows:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828. \tag{1.1}$$

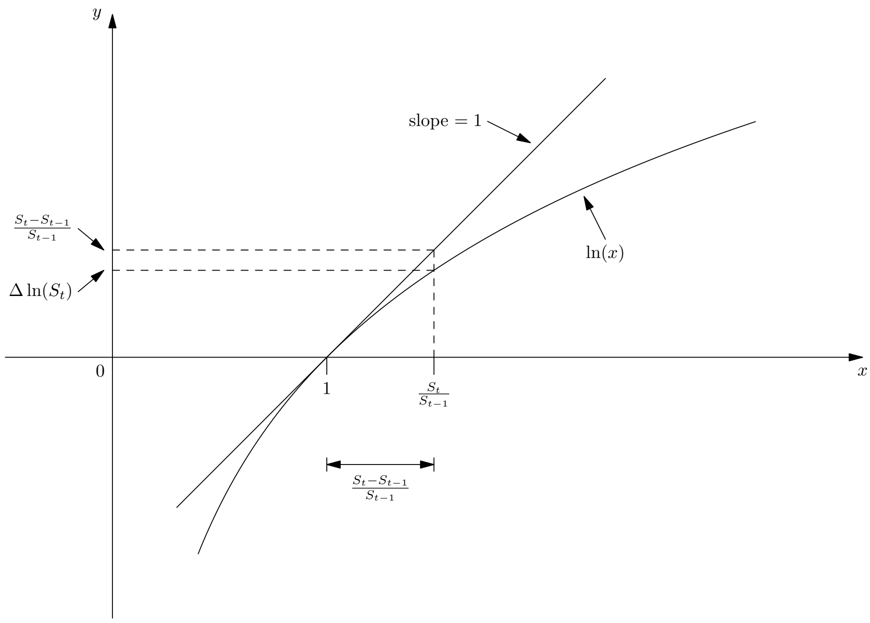

Figure 1.2: Log differencing.

Suppose that $n = 1$. Then, the term of which the limit is taken is:

$$(1 + 1)^1 = 2. \tag{1.2}$$

This can be interpreted as an investment of one euro with a return of 100%.

Now suppose that $n = 2$. Now the term is:

$$(1 + 0.50)^2 = 2.25. \tag{1.3}$$

This can be interpreted as an investment of one euro that is reinvested once during the same period, but with a return of only 50%, or $1/2$, in each subperiod. The final return, also called compound return, is 125%.

We can continue to raise n in this way. For example, if $n = 5$, the term becomes:

$$(1 + 0.20)^5 = 2.48832. \tag{1.4}$$

This amounts to one euro being invested and reinvested five times during a given period, but with a return of only 20%, or $1/5$, in each subperiod. The compound return is 149%.

Now let's suppose we raise n to 100. Then we have:

$$(1 + 0.01)^{100} = 2.70481, \tag{1.5}$$

which is already very close to e. This can be interpreted as an investment of one euro that is invested and reinvested one hundred times, but with a return of only 1%, or $1/100$, in each subperiod. Now, the compound return is approximately 170%.
We see that when n is high, or when $1/n$ is low, we obtain:

$$\left(1+\frac{1}{n}\right)^n \approx e \tag{1.6}$$

$$\Leftrightarrow \quad n\ln\left(1+\frac{1}{n}\right) \approx \ln(e) = 1 \tag{1.7}$$

$$\Leftrightarrow \quad \ln\left(1+\frac{1}{n}\right) \approx \frac{1}{n}. \tag{1.8}$$

If we set $a = 1/n$, we have:

$$\ln(1+a) \approx a. \tag{1.9}$$

Actually, there is another way to derive the same result, which is based on a first-order Taylor series approximation:

$$f(x_0+a) \approx f(x_0) + f'(x)|_{x=x_0}(x-x_0). \tag{1.10}$$

If we let $f(x) = \ln(x)$ and $x_0 = 1$, then we have:

$$\ln(1+a) \approx \ln(1) + \left.\frac{d\ln(x)}{dx}\right|_{x=1} \times (1+a-1) = a, \tag{1.11}$$

since $\ln(1) = 0$ and $\ln(x)/dx = 1/x$.

1.1.3 Interpreting regressions with logarithmic variables

The approximation in equations (1.9) and (1.11) tells us that when a variable x is raised by a small percentage a, its natural logarithm rises by a. This is helpful, for instance when we want to interpret an estimated regression equation. Consider a consumption equation and suppose that C is consumption and Y income and that $c = \ln(C)$ and that $y = \ln(Y)$. Then, depending on how it is formulated, a regression of consumption on income can be interpreted as follows:

$\hat{C} = \hat{\alpha} + \hat{\beta}Y.$	If Y rises by 1 euro, C rises by $\hat{\beta}$ euros.
$\hat{c} = \hat{\alpha} + \hat{\beta}Y.$	If Y rises by 0.01 euros, C rises by $\hat{\beta}$ percent.
$\hat{C} = \hat{\alpha} + \hat{\beta}y.$	If Y rises by 1 percent, C rises by $0.01 \times \hat{\beta}$ euros.
$\hat{c} = \hat{\alpha} + \hat{\beta}y.$	If Y rises by 1 percent, C rises by $\hat{\beta}$ percent.

1.1.4 Log-differencing

Another very useful application of equations (1.9) and (1.11) is log-differencing. This is a method of calculating the growth rate of a variable by first applying the natural logarithm to it and then applying the difference operator.
Let $x_t = \ln(X_t)$ and denote the growth rate of X_t as $\hat{X}_t$.

$$\begin{aligned}\Delta x_t &= x_t - x_{t-1} \\ &= \ln(X_t) - \ln(X_{t-1}) \\ &= \ln\left(\frac{X_t}{X_{t-1}}\right) \\ &= \ln\left(1 + \frac{X_t - X_{t-1}}{X_{t-1}}\right) \\ &\approx \frac{X_t - X_{t-1}}{X_{t-1}} \\ &= \hat{X}_t.\end{aligned} \tag{1.12}$$

To take an example, let's look at the growth rate of real GDP:

1. Definition of real GDP:

$$\bar{Y}_t^{\text{P}} = \frac{Y_t^{\text{P}}}{P_t}. \tag{1.13}$$

2. Taking logarithms:

$$\bar{y}_t^{\text{P}} = y_t^{\text{P}} - p_t. \tag{1.14}$$

3. Applying the difference operator:

$$\begin{aligned}\Delta \bar{y}_t^{\text{P}} &= \Delta y_t^{\text{P}} - \Delta p_t \\ &= g_t^{\text{P}} - \pi_t.\end{aligned} \tag{1.15}$$

We see that the growth rate of real GDP is equal to the growth rate of GDP minus the inflation rate. As another example, we may use log-differencing to derive the rate of appreciation of the real exchange rate:

1. Definition of the real exchange rate:

$$Q_t = \frac{S_t P_t^{\text{H}}}{P_t^{\text{F}}}. \tag{1.16}$$

Economic agent	**Assets**	**Liabilities**
	Assets	Net worth Other liabilities

Table 1.1: General balance sheet.

2. Taking logarithms:

$$q_t = s_t + p_t^{\text{H}} - p_t^{\text{F}}. \tag{1.17}$$

3. Applying the difference operator:

$$\begin{aligned} \Delta q_t &= \Delta s_t + \Delta p_t^{\text{H}} - \Delta p_t^{\text{F}} \\ &= \Delta s_t + \pi_t^{\text{H}} - \pi_t^{\text{F}}. \end{aligned} \tag{1.18}$$

This shows us that the rate of real appreciation is equal to the rate of nominal appreciation plus the domestic inflation rate minus the foreign inflation rate.

1.2 Riqueza

1.2.1 Balances

Households	Assets		Liabilities	
	Money	[1], [2]	Net worth	
	Bonds, shares	[3], [4]	Loans from banks	[5]
	Real assets			
Firms	**Assets**		**Liabilities**	
	Money	[1], [2]	Net worth (book value, equity)	[4]
	Bonds, shares	[3], [4]	Loans from banks	[6]
	Real assets		Corporate bonds	[3]
Government	**Assets**		**Liabilities**	
	Money	[1], [2]	Net worth	
	Bonds, shares	[3], [4]	Government bonds	[3]
	Real assets			
Banks	**Assets**		**Liabilities**	
	Bank notes and coins	[1]	Net worth (book value, equity)	[4]
	Bank reserves	[8]	Bonds	[3]
	Bank loans to households	[5]	Bank deposits	[2]
	Bank loans to firms	[6]		
Central bank	**Assets**		**Liabilities**	
	Domestic credit	[3]	Bank reserves	[8]
	Official reserves	[7]	Bank notes and coins	[1]

Table 1.2: Balance sheet of all economic agents.

Home economy	Assets		Liabilities	
	Bank notes and coins	[1]	Domestic real wealth	
	Bank reserves	[8]	Net external wealth (excl. CB)	
	Real assets		Domestic credit	[3]
	Net foreign assets (excl. CB)			
Home central bank	**Assets**		**Liabilities**	
	Domestic credit	[3]	Bank reserves	[8]
	Domestic official reserves	[7]	Bank notes and coins	[1]

Table 1.3: Balance sheets of the domestic economy and central bank.

Home country	Assets	Liabilities
	Real assets	Domestic real wealth
	Net foreign assets	Net external wealth

Table 1.4: Balance sheet of a country.

Agente económico	Activos	Pasivos
	Activos (A)	Riqueza neta (posición neta, recursos propios, capital propio, patrimonio neto) (E) Otros pasivos (OL)

Table 1.5: Balance general.

Hogares	Activos		Pasivos	
	Dinero (BC, depósitos bancarios)	[1], [2]	Riqueza neta (patrimonio neto)	
	Bonos, acciones	[3], [4]	Préstamos de bancos	[5]
	Activos reales			
Empresas	**Activos**		**Pasivos**	
	Dinero (BC, depósitos bancarios)	[1], [2]	Recursos propios (capital propio, patrimonio neto)	[4]
	Bonos, acciones	[3], [4]	Préstamos de bancos	[6]
	Activos reales		Bones	[3]
Gobierno	**Activos**		**Pasivos**	
	Dinero (BC, depósitos bancarios)	[1], [2]	Posición neta	
	Bonos, acciones	[3], [4]	Bonos	[3]
	Activos reales			
Bancos	**Activos**		**Pasivos**	
	Billetes y monedas	[1]	Recursos propios (capital propio, patrimonio neto)	[4]
	Bank reserves	[8]	Bonos	[3]
	Préstamos a hogares	[5]	Depósitos bancarios	[2]
	Préstamos a empresas	[6]		
Banco central	**Activos**		**Pasivos**	
	Crédito doméstico	[3]	Reservas bancarias	[8]
	Reservas oficiales	[7]	Billetes y monedas (BC)	[1]

Table 1.6: Balances de todos los agentes económicos.

Economía doméstica				
	Billetes y monedas	[1]	Riqueza real doméstica	
	Reservas bancarias	[8]	Activos extranjeros netos (excl. BC)	
	Activos reales		Crédito doméstico	[3]
	Activos extranjeros netos (excl. BC)			
Banco central doméstico	**Activos**		**Pasivos**	
	Activos		Pasivos	
	Crédito doméstico	[3]	Reservas bancarias	[8]

Table 1.7: Balances de los agentes económicos por un lado y el banco central por el otro lado.

País doméstico	Activos	Pasivos
	Activos reales Activos extranjeros netos	Riqueza real doméstica Activos extranjeros netos

Table 1.8: Balance de un país.

Banco comercial	Activos		Pasivos	
Abrir el banco	Dinero	20.0	Recursos propios	20.0
Depósitos	Dinero	100.0	Recursos propios	20.0
			Depósitos	80.0
Préstamos	Préstamos	100.0	Recursos propios	20.0
			Depósitos	80.0
Intereses (10%)	Dinero	10.0	Recursos propios	30.0
	Préstamos	100.0	Depósitos	80.0
Devolver depósitos	Dinero	30.0	Recursos propios	30.0

Table 1.9: Operation of a bank, operación de un banco.

1.2.2 Bancos comerciales

Rentabilidad financiera (return on equity):

$$R^E \times E = R^A \times A \quad \Leftrightarrow \quad R^E = LC \times R^A. \tag{1.19}$$

donde LC es el coeficiente de apalancamiento (leverage coefficient) que está definido de la siguiente manera:

$$\begin{aligned} LC &= \frac{A}{E} \\ &= \frac{A}{A - OL}. \end{aligned} \tag{1.20}$$

La variable OL representa otros pasivos (other liablities).
Entonces, si $R^A = 10\%$, $A = 100$ y $OL = 80$:

$$R^E = 5 \times 10\% = 50\%. \tag{1.21}$$

Apalancamiento (leverage)

Coeficiente de apalancamiento:

$$\text{Coeficiente de apalancamiento} = \frac{\text{Activos}}{\text{Capital}} \tag{1.22}$$

Ejemplo:

- Banco 1: activos 100, capital 20, otros pasivos 80 $\Rightarrow$ apalancamiento 5
- Banco 2: activos 100, capital 5, otros pasivos 95 $\Rightarrow$ apalancamiento 20

Si las cosas van bien y los activos tienen un rendimiento de 10%, los rendimientos sobre el capital propio son:

- Banco 1: 50%
- Banco 2: 200%

Si las cosas van mal y los precios de los activos caen un 10%:

- Banco 1: sigue solvente
- Banco 2: quiebra

Apalancamiento de instituciones financieras en los Estados Unidos en 2007 (según Blanchard and Johnson, 2012):

Bancos comerciales	9,08
Bancos cooperativos	8,07
Sociedades financieras	10,00
Bancos de inversión y fondos de alto riesgo	27,01
Fannie Mae y Freddie Mac	23,05

¿Qué puede hacer el banco 2 para evitar una quiebra?

- Respuesta: reducir apalancamiento (deleveraging)

Dos opciones:

- Obtener más capital (difícil)
- Reducir los activos (no conceder nuevos préstamos, no renovar préstamos existentes, vender acciones etc.)

Efectos del deleveraging:

- Congelación del crédito
- Caída de los mercados de vivienda y de valores

1.2.3 Bancos centrales

Agregados monetarios

Base monetaria:

$$\begin{aligned} MB &= DC + OR && (1.23) \\ &= B_t^{\bar{\mathrm{H}}\mathrm{H}} + B_t^{\bar{\mathrm{H}}\mathrm{F}} && (1.24) \\ &= BR + BC. && (1.25) \end{aligned}$$

M0:

$$M0 = BC. \quad (1.26)$$

M2:

$$M2 = BC + BD. \quad (1.27)$$

Notación:

$$
\begin{aligned}
MB &= \text{money base (monetary base, base money, high-powered money), base monetaria,} && (1.28)\\
DC &= \text{domestic credit, crédito doméstico,} && (1.29)\\
OR &= \text{official reserves, reservas oficiales,} && (1.30)\\
BR &= \text{bank reserves, reservas bancarias,} && (1.31)\\
BC &= \text{bank notes and coins,} && (1.32)\\
BD &= \text{bank deposits.} && (1.33)
\end{aligned}
$$

Creación de dinero

$$R_{t+1} = \frac{P_{t+1}^B - P_t^B + C_{t+1}^B}{P_t^B} = \frac{P_{t+1}^B + C_{t+1}^B}{P_t^B} - 1 \quad \Leftrightarrow \quad P_t^B = \frac{P_{t+1}^B + C_{t+1}^B}{1 + R_{t+1}}, \tag{1.34}$$

donde C_{t+1}^B es el cupón del bono.

Política monetaria:

$$R_{t+1}\downarrow \quad \Leftrightarrow \quad P_t^B \uparrow \quad \Leftrightarrow \quad B_t^{\bar{\mathrm{H}}\mathrm{H}} \uparrow \quad \Leftrightarrow \quad M_t \uparrow \quad \Leftrightarrow \quad P_t \uparrow \quad \Leftrightarrow \quad \pi_t^{\mathrm{H}} = \frac{P_t^{\mathrm{H}} - P_{t-1}^{\mathrm{H}}}{P_{t-1}^{\mathrm{H}}} \uparrow, \tag{1.35}$$

$$R_{t+1}\uparrow \quad \Leftrightarrow \quad P_t^B \downarrow \quad \Leftrightarrow \quad B_t^{\bar{\mathrm{H}}\mathrm{H}} \downarrow \quad \Leftrightarrow \quad M_t \downarrow \quad \Leftrightarrow \quad P_t \downarrow \quad \Leftrightarrow \quad \pi_t^{\mathrm{H}} = \frac{P_t^{\mathrm{H}} - P_{t-1}^{\mathrm{H}}}{P_{t-1}^{\mathrm{H}}} \downarrow. \tag{1.36}$$

Quantitative easing (expansión cuantitativa):

$$B_t^{\bar{\mathrm{H}}\mathrm{H}} \uparrow \quad \Leftrightarrow \quad M_t \uparrow \quad \Leftrightarrow \quad P_t \uparrow \quad \Leftrightarrow \quad \pi_t^{\mathrm{H}} = \frac{P_t^{\mathrm{H}} - P_{t-1}^{\mathrm{H}}}{P_{t-1}^{\mathrm{H}}} \uparrow, \tag{1.37}$$

$$R_{t+1} = 0 \quad \Leftrightarrow \quad P_t^B = P_{t+1}^B + C_{t+1}^B. \tag{1.38}$$

1.3 Wealth across countries

Country	Total wealth (trillion USD)	Real wealth (trillion USD)	IIP (trillion USD)	GDP (trillion USD)	Adult population (m)
World	317.084	317.084	0.000	87.265	5025.085
United States	98.150	108.079	-9.929	21.344	242.972
China	51.874	50.127	1.747	14.216	1085.003
India	12.833	13.212	-0.379	3.050	1392.058
Japan	23.884	20.817	3.067	5.176	105.108
Germany	14.499	12.702	1.797	3.963	67.470
United Kingdom	14.209	14.578	-0.369	2.829	50.919
France	13.883	14.253	-0.370	2.761	49.478
Italy	10.569	10.895	-0.326	2.025	48.527
Spain	7.152	8.084	-0.932	1.429	37.410
Netherlands	3.357	2.799	0.558	0.914	13.260
Belgium	2.776	2.556	0.220	0.512	8.869
Morocco	0.216	0.291	-0.075	0.121	23.218

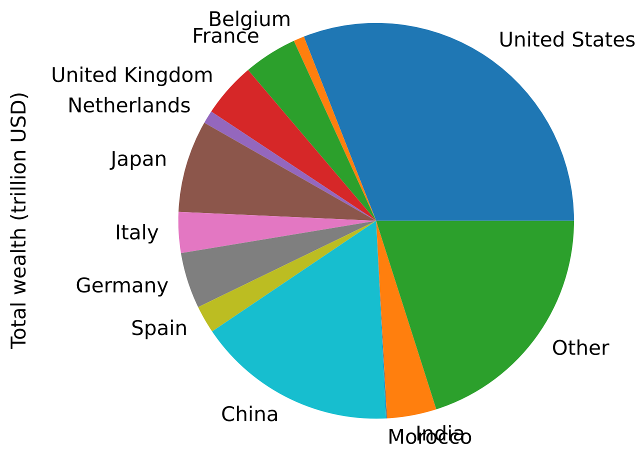

Figure 1.3: Total wealth of selected countries.

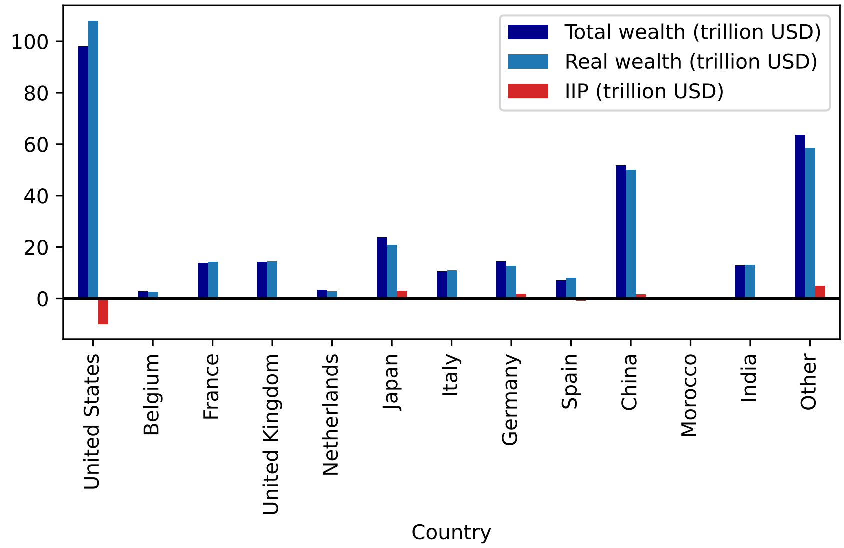

Figure 1.4: Total wealth, real wealth and the net international investment position (NIIP).

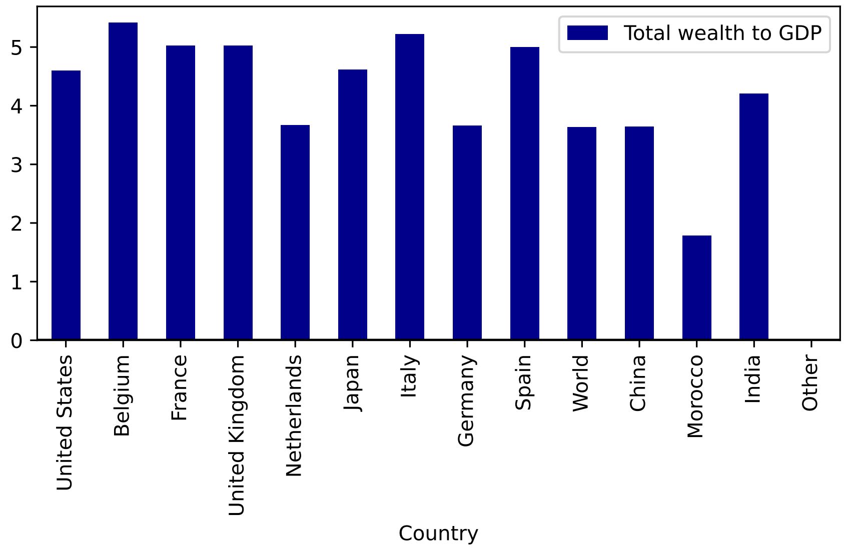

Figure 1.5: Total-wealth-to-GDP ratio.

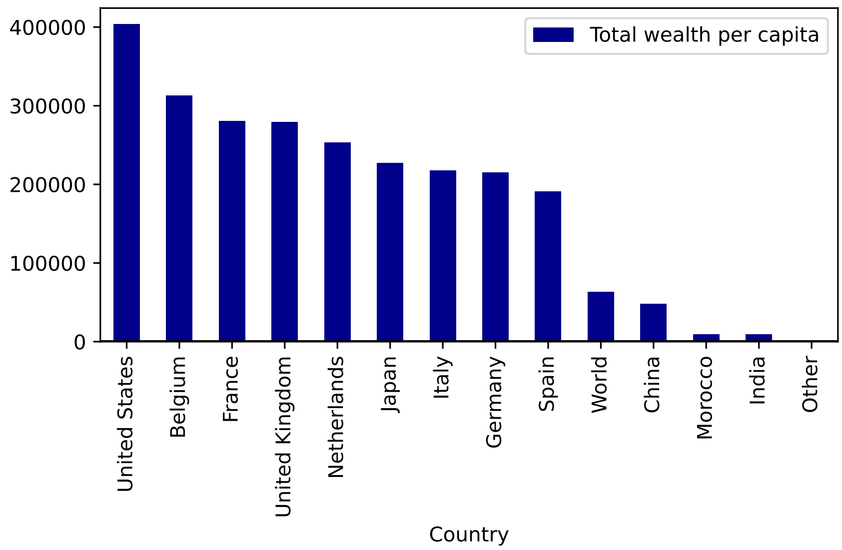

Figure 1.6: Total wealth per capita.

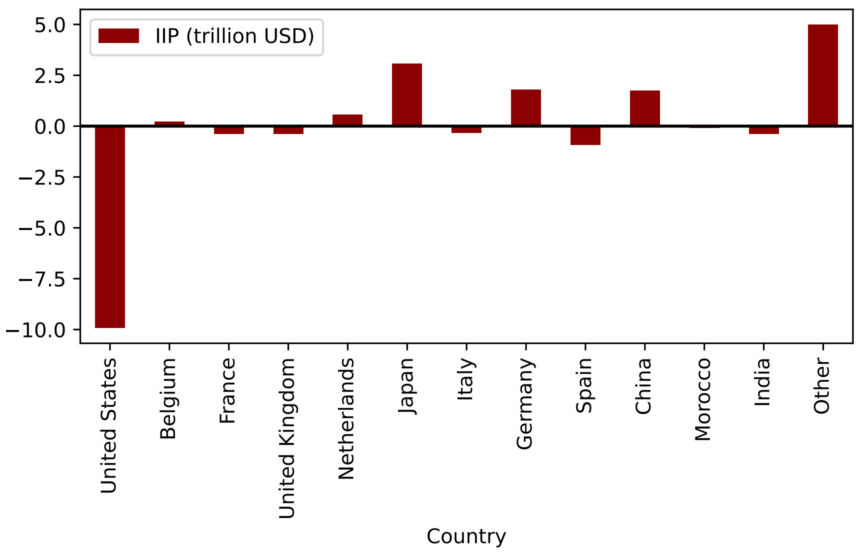

Figure 1.7: Net international investment position.

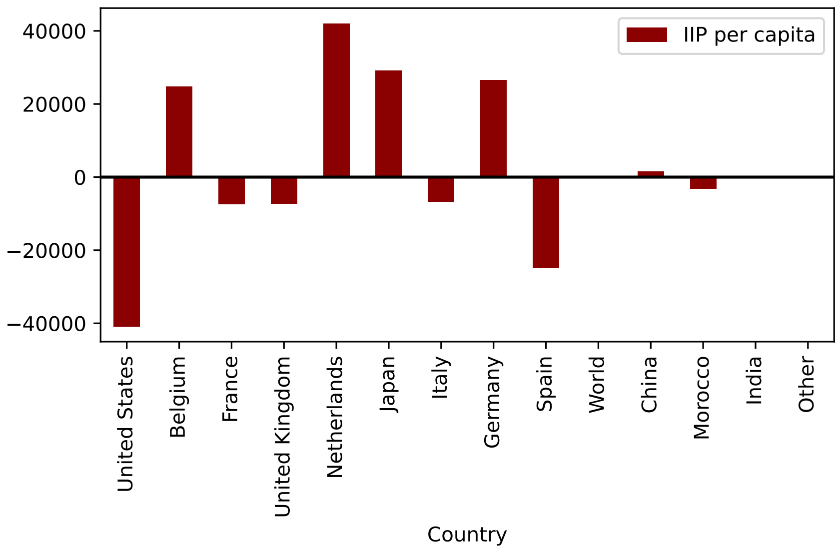

Figure 1.8: Net international investment position per capita.

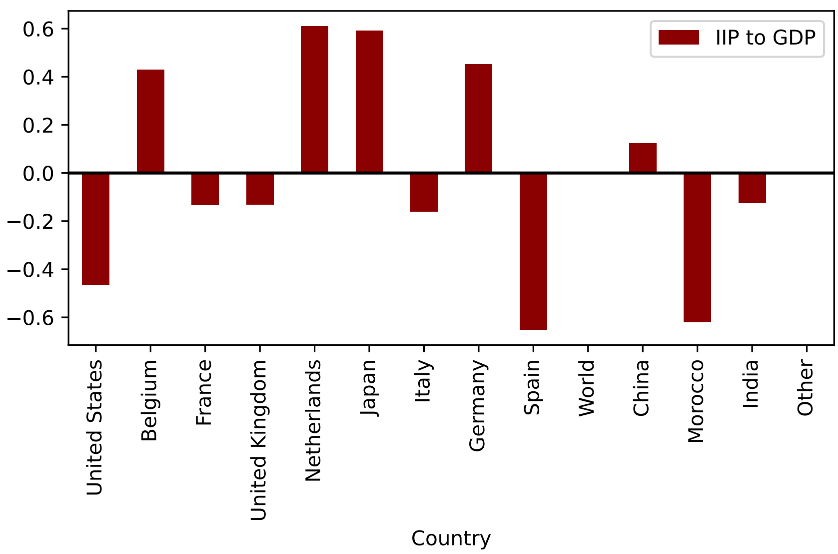

Figure 1.9: Ratio of net international investment position to GDP.

1.4 Balance of payments

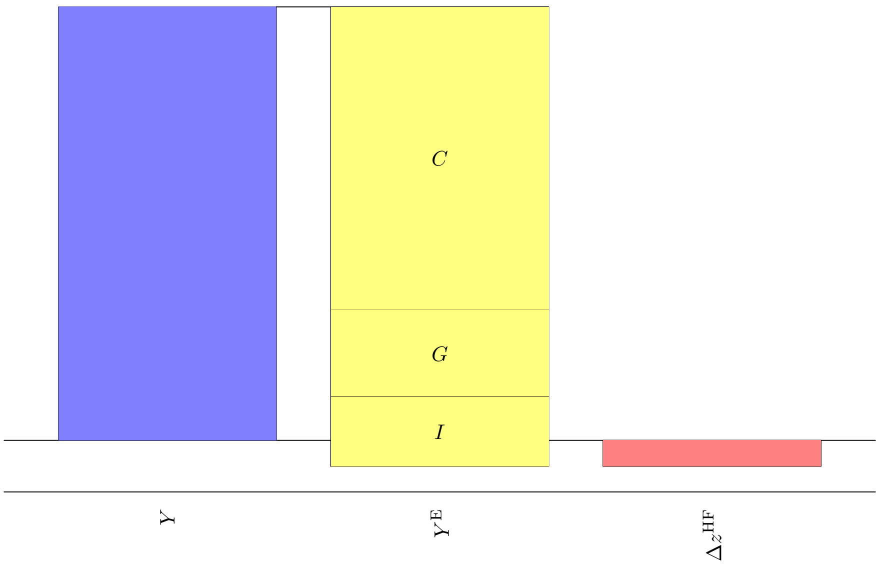

Figure 1.10: The current account as gross national disposable income minus gross national expenditure.

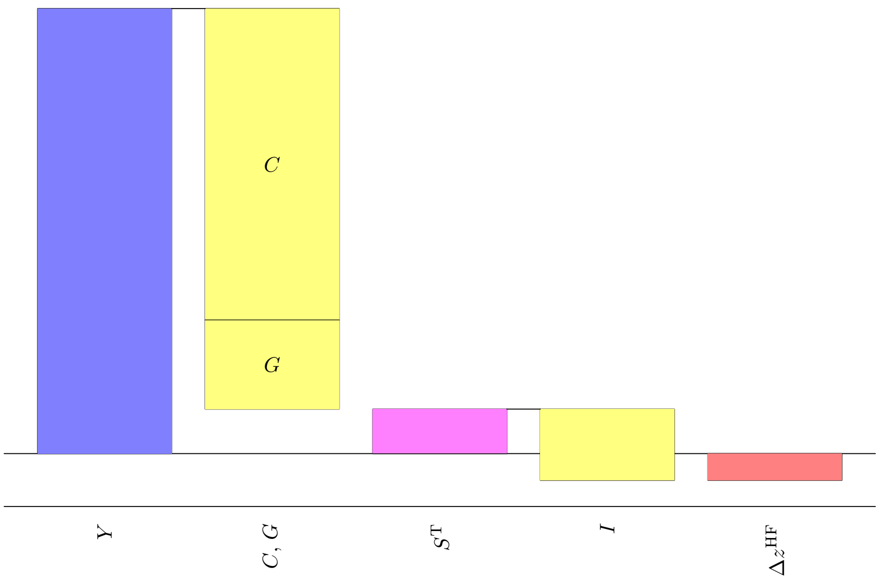

Figure 1.11: The current account as saving minus investment.

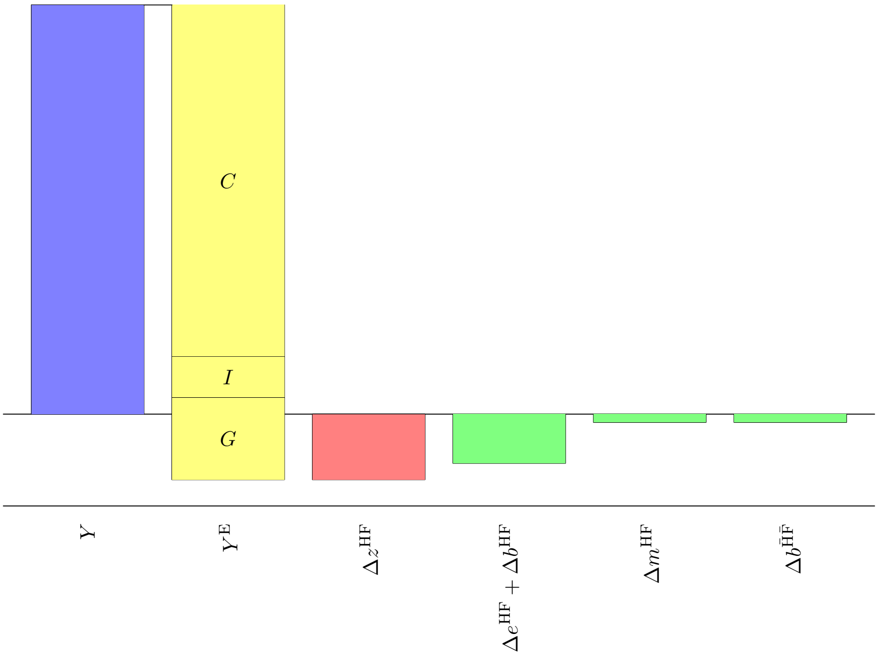

Figure 1.12: Current account deficit associated with capital inflows, money outflows and reserve losses.

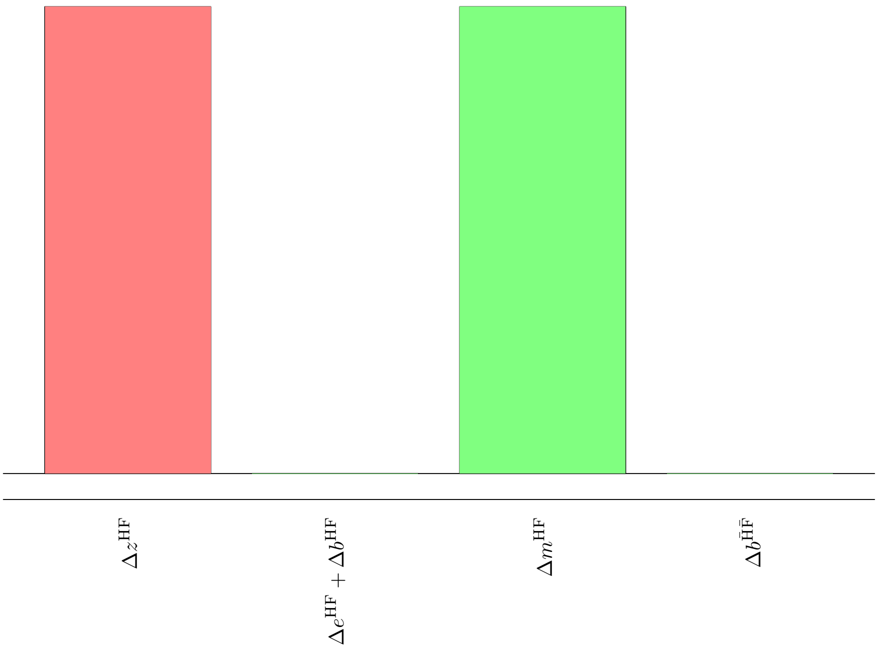

Figure 1.13: Current account surplus associated with money inflows.

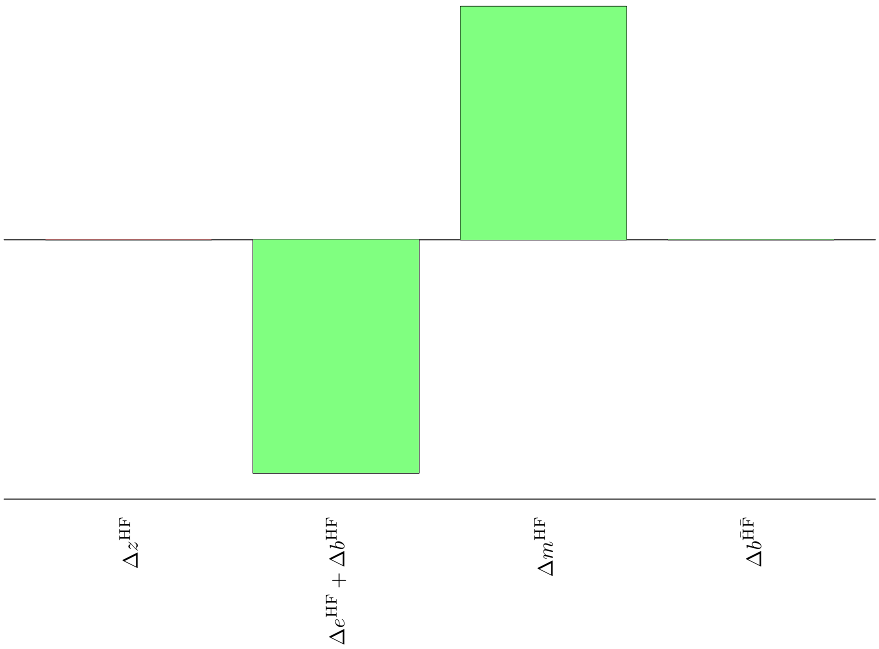

Figure 1.14: Capital inflows associated with money inflows.

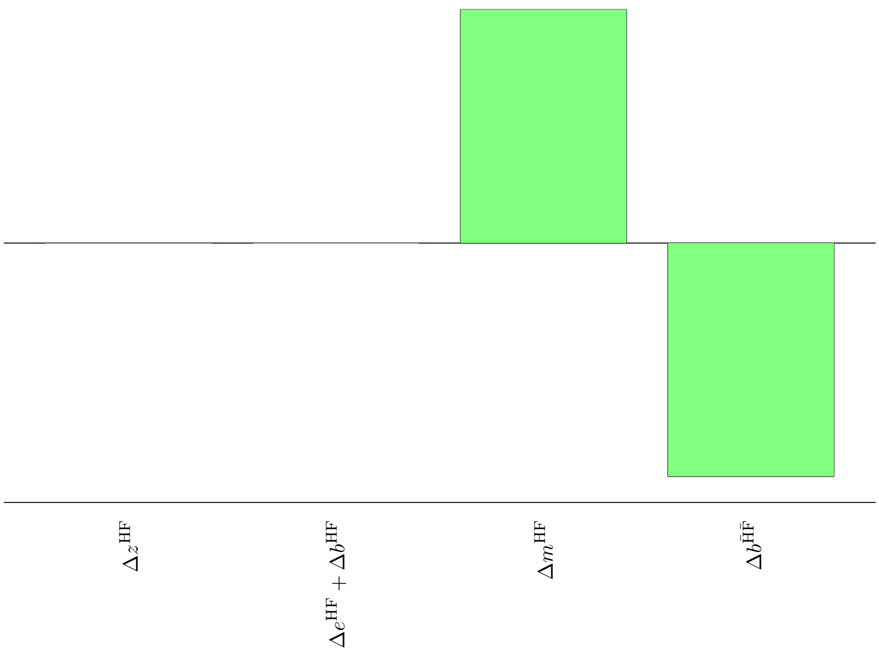

Figure 1.15: Sales of official reserves associated with money inflows.

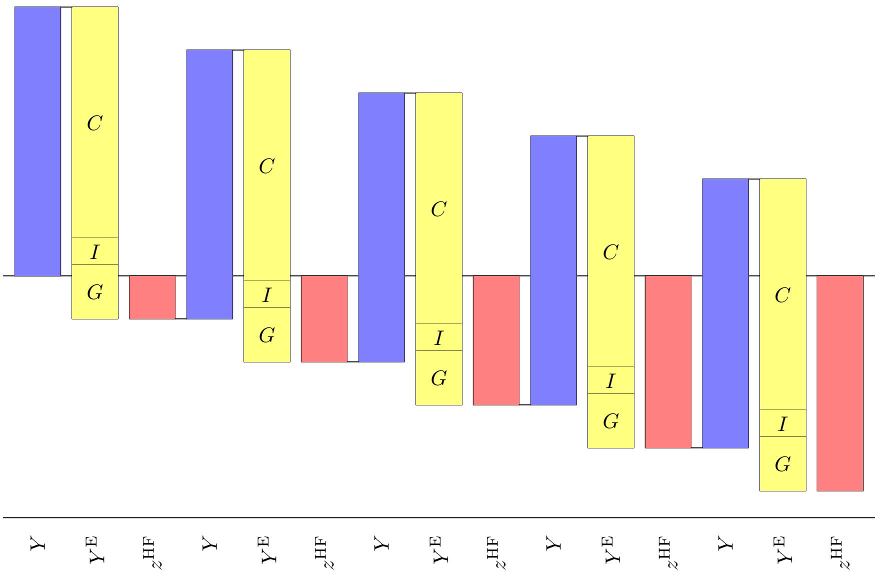

Figure 1.16: The current account and the international investment position.

1.5 Volatilities of national income components

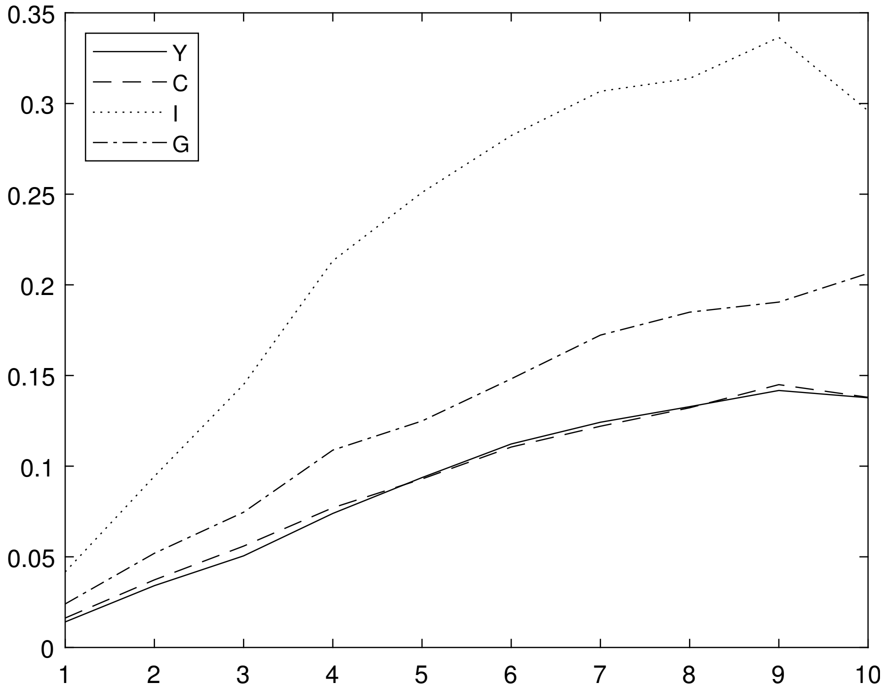

Figure 1.17: Volatilities of national income components at different horizons: national income (Y), private consumption (C), investment (I) and government spending (G). Horizons are measured in years. Source: International Financial Statistics (IMF), author's calculations.

The volatility of a given national income component, X, is measured as:

$$\begin{aligned}
\mathrm{Var}(\Delta_h x_t) &= \mathrm{Var}(x_t - x_{t-h}) \\
&= \mathrm{Var}(\ln(X_t) - \ln(X_{t-h})) \\
&= \mathrm{Var}\left(\ln\left(\frac{X_t}{X_{t-h}}\right)\right) \\
&= \mathrm{Var}\left(\ln\left(1 + \frac{X_t - X_{t-h}}{X_{t-h}}\right)\right) \\
&\approx \mathrm{Var}\left(\frac{X_t - X_{t-h}}{X_{t-h}}\right).
\end{aligned} \tag{1.39}$$

where h is the horizon.

Recall that $\ln(1+a) \approx a$ if a is small.

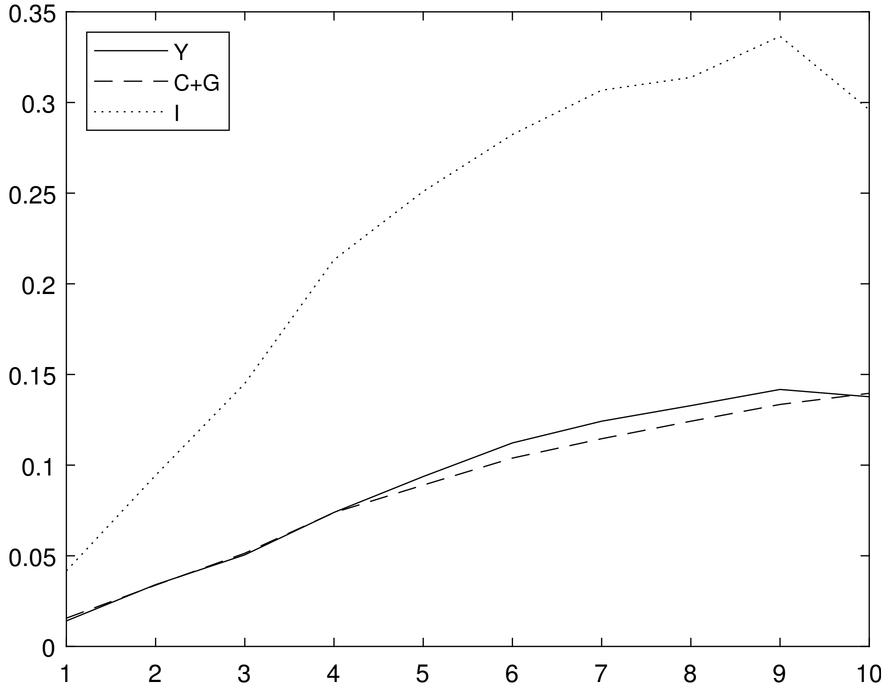

Figure 1.18: Volatilities of national income components at different horizons: national income (Y), private and public consumption ($C + G$) and investment (I). Horizons are measured in years. Source: International Financial Statistics (IMF), author's calculations.

1.6 Currency flow model

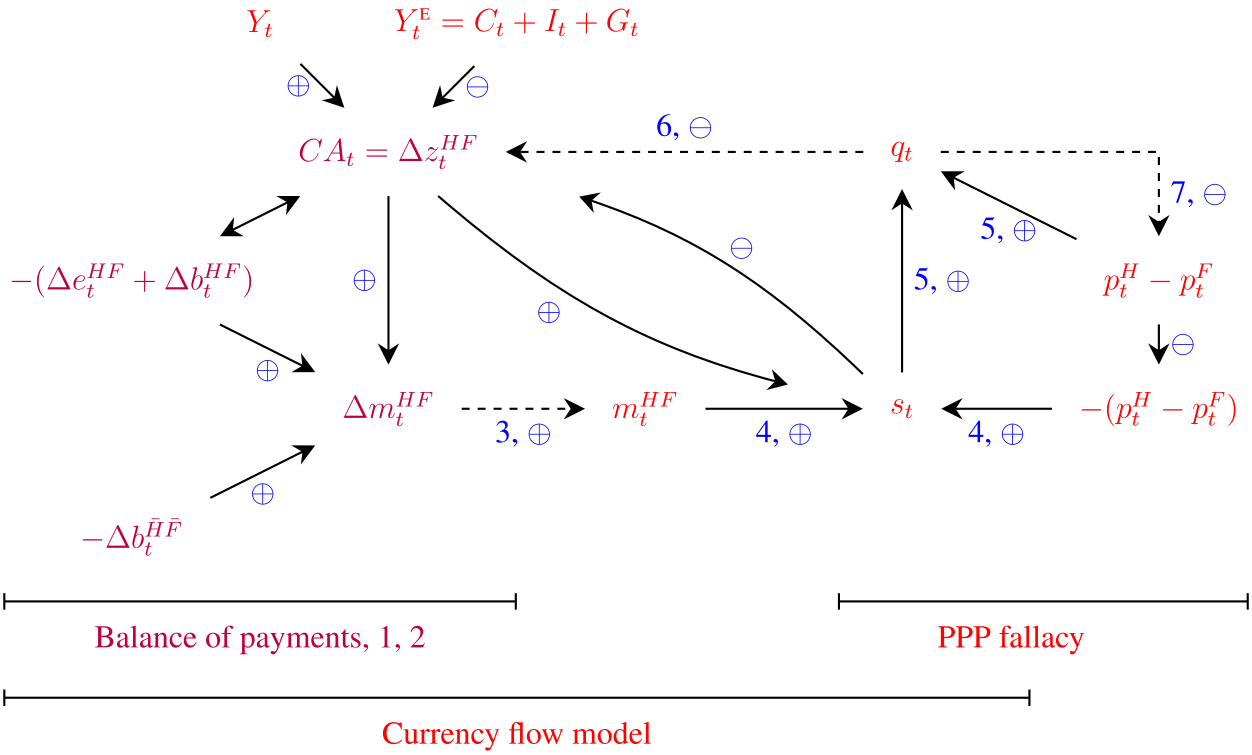

Figure 1.19: Currency flow model

Net money inflows as sum as current account balance, net ”capital inflows” and net sales of official reserves:

$$\begin{aligned} CA_t &= \Delta z_t^{\mathrm{HF}} \\ &= \Delta e_t^{\mathrm{HF}} + \Delta b_t^{\mathrm{HF}} + \Delta m_t^{\mathrm{HF}} + \Delta b_t^{\bar{\mathrm{H}}\bar{\mathrm{F}}}, \end{aligned} \tag{1}$$

$$\Leftrightarrow \quad \Delta m_t^{\mathrm{HF}} = CA_t - (\Delta e_t^{\mathrm{HF}} + \Delta b_t^{\mathrm{HF}}) - \Delta b_t^{\bar{\mathrm{H}}\bar{\mathrm{F}}}. \tag{2}$$

Accumulation of net foreign money holdings:

$$m_t^{\mathrm{HF}} = m_{t-1}^{\mathrm{HF}} + \Delta m_t^{\mathrm{HF}}. \tag{3}$$

Determination of nominal exchange rate in currency flow model:

$$s_t = -(p_t^{\mathrm{H}} - p_t^{\mathrm{F}}) + \xi m_t^{\mathrm{HF}}. \tag{4}$$

Definition of real exchange rate:

$$q_t = s_t + p_t^{\text{H}} - p_t^{\text{F}}. \tag{5}$$

Effect of real exchange rate on net exports:

$$\Delta CA_t = -\phi q_{t-1}. \tag{6}$$

Good-market arbitrage:

$$\Delta(p_t^{\text{H}} - p_t^{\text{F}}) = -\psi q_t. \tag{7}$$

1.7 Over- and undervaluation of the nominal exchange rate

The over- or undervaluation Ω of the nominal exchange rate can be computed as follows:

$$\Omega = \frac{S^{\text{actual}}}{S^{\text{hypothetical}}} - 1. \tag{1.40}$$

For example, if $S^{\text{actual}} = 2.50$ and $S^{\text{hypothetical}} = 2.00$, then the nominal exchange rate is 25% overvalued:

$$\Omega = \frac{2.50}{2.00} - 1 = 1.25 - 1 = +0.25 = +25\%. \tag{1.41}$$

In other words, our currency is worth 2.50 units of the foreign currency rather than 2.00 units, which is what we consider adequate, so our currency has a higher value (in terms of the foreign currency) than it should have.

To decide whether an exchange rate is over- or undervalued, one thus has to first decide what is an ”adequate” level of the exchange rate.

One way to proceed is to select an ”adequate” level for the *real* exchange rate and then compute the *nominal* exchange rate that is compatible with this real exchange rate. We call this the hypothetical level of the nominal exchange rate (the level of the nominal exchange rate that is compatible with the previously selected ”adequate” level of the real exchange rate) and denote it as $S^{\text{hypothetical}}$.

To decide on the ”adequate” level of the real exchange rate, we can resort to the theory of purchasing power parity.

1.7.1 Example 1: $Q = 1$ (absolute PPP)

Let’s suppose that $Q = 1$, which implies absolute purchasing power parity. Let’s denote the hypothetical nominal exchange rate in this case as $S^{Q=1}$. Then:

$$Q = \frac{S^{Q=1} P^{\text{H}}}{P^{\text{F}}} = 1 \quad \Leftrightarrow \quad S^{Q=1} = \frac{P^{\text{F}}}{P^{\text{H}}}. \tag{1.42}$$

The over- or undervaluation Ω in this case becomes:

$$\Omega = \frac{S^{\text{actual}}}{S^{\text{hypothetical}}} - 1 = \frac{S}{\frac{P^{\text{F}}}{P^{\text{H}}}} - 1 = Q - 1. \tag{1.43}$$

Note that Ω can be inferred directly from Q.

1.7.2 Example 2: $Q = \bar{Q}$ (relative PPP)

Let's now consider the more general case where Q is constant at $Q = \bar{Q}$, which is the case of relative purchasing power parity. Let's denote the hypothetical nominal exchange rate in this case as $S^{Q=\bar{Q}}$. Then:

$$Q = \frac{S^{Q=\bar{Q}} P^{\text{H}}}{P^{\text{F}}} = \bar{Q} \quad \Leftrightarrow \quad S^{Q=\bar{Q}} = \bar{Q} \times \frac{P^{\text{F}}}{P^{\text{H}}}. \tag{1.44}$$

The over- or undervaluation Ω in this case becomes:

$$\Omega = \frac{S^{\text{actual}}}{S^{\text{hypothetical}}} - 1 = \frac{S}{\bar{Q} \times \frac{P^{\text{F}}}{P^{\text{H}}}} - 1 = \frac{Q}{\bar{Q}} - 1. \tag{1.45}$$

Note that Ω can be inferred directly from Q and $\bar{Q}$.
Of course, example 2 is just a special case of example 1 with $\bar{Q} = 1$.

1.8 Logarithm to base b

$$\begin{aligned}
\log_b(X) &= \log_b\left(e^{\ln(X)}\right) && (1.46)\\
&= \ln(X)\log_b(e) && (1.47)\\
&= \frac{\ln(X)}{\ln(b)}\ln(b)\log_b(e) && (1.48)\\
&= \frac{\ln(X)}{\ln(b)}\log_b\left(e^{\ln(b)}\right) && (1.49)\\
&= \frac{\ln(X)}{\ln(b)}\log_b(b) && (1.50)\\
&= \frac{\ln(X)}{\ln(b)}. && (1.51)
\end{aligned}$$

For example:

$$\begin{aligned}
\log_2 &= \frac{\ln(X)}{\ln(2)}, && (1.52)\\
\log_{10} &= \frac{\ln(X)}{\ln(10)}. && (1.53)
\end{aligned}$$

1.9 Balassa-Samuelson effect

1.9.1 Wage and price setting

Maximization problem for firms:

$$\max_L PY(L) - WL. \tag{1.54}$$

First-order condition for maximum:

$$PY_L(L) - W = 0 \quad \Leftrightarrow \quad W = PY_L, \tag{1.55}$$

where

$$Y_L = \frac{dY(L)}{dL}. \tag{1.56}$$

In logarithms:

$$\begin{aligned} & w &&= p + y_L && (1.57)\\ \Leftrightarrow\quad & w - p &&= y_L && (1.58)\\ \Leftrightarrow\quad & p &&= w - y_L. && (1.59) \end{aligned}$$

Note that the real wage, $w - p$, is equal to the marginal productivity of labour, y_L, which implies that the price level, p, is inversely related to the y_L.

Let there be two countries and two sectors:

- H: home country
- F: foreign country
- T: traded good sector
- N: non-traded good sector

In each country, for workers to have an incentive to work in both industries, wages in both sectors have to be equal:

$$w^{\mathrm{H}} = w^{\mathrm{T,H}} = w^{\mathrm{N,H}}, \tag{1.60}$$

$$w^{\mathrm{F}} = w^{\mathrm{T,F}} = w^{\mathrm{N,F}}, \tag{1.61}$$

Hence we have four price-setting equations:

$$p^{\text{T,H}} = w^{\text{H}} - y_L^{\text{T,H}}, \tag{1.62}$$
$$p^{\text{N,H}} = w^{\text{H}} - y_L^{\text{N,H}}, \tag{1.63}$$
$$p^{\text{T,F}} = w^{\text{F}} - y_L^{\text{T,F}}, \tag{1.64}$$
$$p^{\text{N,F}} = w^{\text{F}} - y_L^{\text{N,F}}. \tag{1.65}$$

The overall price levels in both countries are given by the following two equations:

$$p^{\text{H}} = (1-\alpha)p^{\text{T,H}} + \alpha p^{\text{N,H}} \tag{1.66}$$
$$= p^{\text{T,H}} + \alpha(p^{\text{N,H}} - p^{\text{T,H}}), \tag{1.67}$$
$$p^{\text{F}} = (1-\alpha)p^{\text{T,F}} + \alpha p^{\text{N,F}} \tag{1.68}$$
$$= p^{\text{T,F}} + \alpha(p^{\text{N,F}} - p^{\text{T,F}}). \tag{1.69}$$

The difference of the domestic and foreign price levels:

$$p^{\text{H}} - p^{\text{F}} = p^{\text{T,H}} - p^{\text{T,F}} + \alpha[(p^{\text{N,H}} - p^{\text{T,H}}) - (p^{\text{N,F}} - p^{\text{T,F}})] \tag{1.70}$$
$$= p^{\text{T,H}} - p^{\text{T,F}} + \alpha \underbrace{[\underbrace{(y_L^{\text{T,H}} - y_L^{\text{N,H}})}_{\text{high}} - \underbrace{(y_L^{\text{T,F}} - y_L^{\text{N,F}})}_{\text{low}}]}_{>0} \tag{1.71}$$

Note that if the home country is more productive, and thus richer, than the foreign country, it is likely that this productivity advantage is concentrated in the production of traded goods. This why it is reasonable to assume that $y_L^{\text{T,H}} - y_L^{\text{N,H}}$ is higher than $y_L^{\text{T,F}} - y_L^{\text{N,F}}$, which implies that the term in the square brackets of equation (1.71) is positive (as indicated).

1.9.2 Balassa-Samuelson hypothesis

The Balassa-Samuelson hypothesis is a theory that aims to explain the Balassa-Samuelson effect, that is, the empirical observation that prices tend to be higher in richer countries and lower in poorer countries.

The Balassa-Samuelson hypothesis is based on the assumption that traded good prices, once converted to the same currency, cannot differ between countries. Thus the traded good real exchange rate is equal to zero:

$$q^{\text{T}} = s + p^{\text{T,H}} - p^{\text{T,F}} = 0. \tag{1.72}$$

Note that this implies that the nominal exchange rate has to be equal to the ratio of the purchasing powers of the domestic and foreign currencies:

$$s = -(p^{\text{T,H}} - p^{\text{T,F}}). \tag{1.73}$$

In this case, the real exchange rate reflects the gap between the relative productivities of the traded good sectors of both countries:

$$\begin{aligned} q &= s + p^{\text{H}} - p^{\text{F}} && (1.74)\\ &= s + p^{\text{T,H}} - p^{\text{T,F}} + \alpha[(p^{\text{N,H}} - p^{\text{N,F}}) - (p^{\text{T,H}} - p^{\text{T,F}})] && (1.75)\\ &= \qquad \alpha[(y_L^{\text{T,H}} - y_L^{\text{N,H}}) - (y_L^{\text{T,F}} - y_L^{\text{N,F}})] && (1.76)\\ &> 0. && (1.77) \end{aligned}$$

1.9.3 The currency flow model

Currency market pressure versus the Balassa-Samuelson hypothesis

Now suppose that the nominal exchange rate, instead of being given by equation (1.73), is determined as in the currency flow model:

$$s = -(p^{\text{T,H}} - p^{\text{T,F}}) + \xi m^{\text{HF}}. \tag{1.78}$$

In this case the traded good real exchange rate does not have to equal zero any more:

$$\begin{aligned} q^{\text{T}} &= s + p^{\text{T,H}} - p^{\text{T,F}} && (1.79)\\ &= -(p^{\text{T,H}} - p^{\text{T,F}}) + \xi m^{\text{HF}} + p^{\text{T,H}} - p^{\text{T,F}} && (1.80)\\ &= \xi m^{\text{HF}} && (1.81)\\ &= \text{CMP}, && (1.82) \end{aligned}$$

where CMP represents currency market pressure. In other words, traded good prices can differ between countries.

The real exchange rate is now given by a weighted average of the currency market pressure on the one hand and the gap between the relative traded good productivities in both countries on the other hand:

$$\begin{aligned} q &= s + p^{\text{H}} - p^{\text{F}} && (1.83)\\ &= s + p^{\text{T,H}} - p^{\text{T,F}} + \alpha[(p^{\text{N,H}} - p^{\text{N,F}}) - (p^{\text{T,H}} - p^{\text{T,F}})] && (1.84) \end{aligned}$$

$$= q^{\mathrm{T}} \qquad + \alpha[(y_L^{\mathrm{T,H}} - y_L^{\mathrm{N,H}}) - (y_L^{\mathrm{T,F}} - y_L^{\mathrm{N,F}})] \tag{1.85}$$
$$= \xi m^{\mathrm{HF}} \qquad + \alpha[(y_L^{\mathrm{T,H}} - y_L^{\mathrm{N,H}}) - (y_L^{\mathrm{T,F}} - y_L^{\mathrm{N,F}})]. \tag{1.86}$$

What if traded good and non-traded good productivities do not matter at all?

Note that in determining the nominal exchange rate, equation (1.126) expressed the domestic and foreign currencies' purchasing powers in terms of traded good prices (rather than overall price levels).

However, if we measure the purchasing powers of the domestic and foreign currencies in terms of overall price levels, we find that the real exchange rate depends exclusively on currency market pressure:

$$s = -(p^{\mathrm{H}} - p^{\mathrm{F}}) + \xi m^{\mathrm{HF}}, \tag{1.87}$$
$$q = s + p^{\mathrm{H}} - p^{\mathrm{F}} \tag{1.88}$$
$$= -(p^{\mathrm{H}} - p^{\mathrm{F}}) + \xi m^{\mathrm{HF}} + p^{\mathrm{H}} - p^{\mathrm{F}} \tag{1.89}$$
$$= \xi m^{\mathrm{HF}}. \tag{1.90}$$

1.9.4 Prices in the traded good and non-traded good sectors

Note that for the derivation of the real exchange rate in the Balassa-Samuelson model, we did not need to calculate the price levels in the traded and non-traded good sectors. All we needed to know was the difference between traded and non-traded good prices, which is inversely related to the difference between the marginal productivities in both sectors.

Although we thus do not need to know the price levels in the traded and non-traded good sectors, it is nevertheless possible to compute them. Let's look at the home country, for example. Here, we have three equations: one for the traded good price level, $p^{\mathrm{T,H}}$, one for the non-traded good price level, $p^{\mathrm{N,H}}$, and one for the overall price level, p^{H}, which is a (geometric) weighted average of the two aforementioned price levels:

$$p^{\mathrm{T,H}} = w^{\mathrm{H}} - y_L^{\mathrm{T,H}}, \tag{1.91}$$
$$p^{\mathrm{N,H}} = w^{\mathrm{H}} - y_L^{\mathrm{N,H}}, \tag{1.92}$$
$$p^{\mathrm{H}} = (1-\alpha)p^{\mathrm{T,H}} + \alpha p^{\mathrm{N,H}}. \tag{1.93}$$

We assume that the productivities in the traded and non-traded good sectors are exogenously given by the technology employed in both sectors. Moreover, we can regard the overall price level as exogenous, too, since it can be controlled by the central bank through its monetary policy. Therefore, equations (1.91) to (1.93) represent three equations, which can be solved for the three

endogenous variables: the nominal wage, w^{H}, the traded good price level, $p^{\text{T,H}}$, and the non-traded good price level, $p^{\text{N,H}}$.
The nominal wage, w^{H}, can be derived as follows:

$$\begin{aligned} p^{\text{H}} &= (1-\alpha)p^{\text{T,H}} + \alpha p^{\text{N,H}} && (1.94)\\ &= (1-\alpha)(w^{\text{H}} - y_L^{\text{T,H}}) + \alpha(w^{\text{H}} - y_L^{\text{N,H}}) && (1.95)\\ &= w^{\text{H}} - (1-\alpha)y_L^{\text{T,H}} - \alpha y_L^{\text{N,H}}, && (1.96)\\ \Leftrightarrow \quad w^{\text{H}} &= p^{\text{H}} + (1-\alpha)y_L^{\text{T,H}} + \alpha y_L^{\text{N,H}}. && (1.97) \end{aligned}$$

We see that the real wage in the home country, $w^{\text{H}} - p^{\text{H}}$, is the weighted average of the marginal productivities of its traded and non-traded good sectors.
The price levels of the traded and non-traded goods, $p^{\text{T,H}}$ and $p^{\text{N,H}}$, can be computed as follows:

$$\begin{aligned} p^{\text{T,H}} &= w^{\text{H}} - y_L^{\text{T,H}}, && (1.98)\\ &= p^{\text{H}} - \alpha(y_L^{\text{T,H}} - y_L^{\text{N,H}}), && (1.99)\\ p^{\text{N,H}} &= w^{\text{H}} - y_L^{\text{N,H}}, && (1.100)\\ &= p^{\text{H}} + (1-\alpha)(y_L^{\text{T,H}} - y_L^{\text{N,H}}). && (1.101) \end{aligned}$$

Hence we see that the higher is the relative productivity in the traded good sector, the lower are the traded-good prices, and the higher are the non-traded good prices, compared to the overall price level set by the domestic central bank.

1.10 El efecto Balassa-Samuelson

1.10.1 Fijación de salarios y precios

El problema de maximización de las empresas:

$$\max_{L} PY(L) - WL. \tag{1.102}$$

La condición de primer orden para el maximum:

$$PY_L(L) - W = 0 \quad \Leftrightarrow \quad W = PY_L, \tag{1.103}$$

donde

$$Y_L = \frac{dY(L)}{dL}. \tag{1.104}$$

En logaritmos:

$$\begin{aligned} & w &&= p + y_L && (1.105)\\ \Leftrightarrow\quad & w - p &&= y_L && (1.106)\\ \Leftrightarrow\quad & p &&= w - y_L. && (1.107)\end{aligned}$$

Observa que el salario real, $w - p$, es igual a la productividad marginal del trabajo, y_L, lo cual implica que el nivel de precios, p, está inversamente relacionado con y_L.
Suponemos que hay dos países y dos sectores:

- H: país doméstico (Home)
- F: país extranjero (Foreign)
- T: sector de bienes comercializables
- N: sector de bienes no comercializables

En cada país, los salarios en ambos sectores tienen que ser iguales para que los trabajadores tengan un incentivo de trabajar en ambas industrias:

$$w^{\mathrm{H}} = w^{\mathrm{T,H}} = w^{\mathrm{N,H}}, \tag{1.108}$$
$$w^{\mathrm{F}} = w^{\mathrm{T,F}} = w^{\mathrm{N,F}}, \tag{1.109}$$

Entonces tenemos cuatro ecuaciones de fijación de precios:

$$p^{\text{T,H}} = w^{\text{H}} - y_L^{\text{T,H}}, \tag{1.110}$$
$$p^{\text{N,H}} = w^{\text{H}} - y_L^{\text{N,H}}, \tag{1.111}$$
$$p^{\text{T,F}} = w^{\text{F}} - y_L^{\text{T,F}}, \tag{1.112}$$
$$p^{\text{N,F}} = w^{\text{F}} - y_L^{\text{N,F}}. \tag{1.113}$$

Los niveles de precios en ambos países vienen dados por las siguientes dos ecuaciones:

$$p^{\text{H}} = (1-\alpha)p^{\text{T,H}} + \alpha p^{\text{N,H}} \tag{1.114}$$
$$= p^{\text{T,H}} + \alpha(p^{\text{N,H}} - p^{\text{T,H}}), \tag{1.115}$$
$$p^{\text{F}} = (1-\alpha)p^{\text{T,F}} + \alpha p^{\text{N,F}} \tag{1.116}$$
$$= p^{\text{T,F}} + \alpha(p^{\text{N,F}} - p^{\text{T,F}}). \tag{1.117}$$

La diferencia entre los niveles de precios doméstico y extranjero:

$$p^{\text{H}} - p^{\text{F}} = p^{\text{T,H}} - p^{\text{T,F}} + \alpha[(p^{\text{N,H}} - p^{\text{T,H}}) - (p^{\text{N,F}} - p^{\text{T,F}})] \tag{1.118}$$
$$= p^{\text{T,H}} - p^{\text{T,F}} + \alpha \underbrace{[\underbrace{(y_L^{\text{T,H}} - y_L^{\text{N,H}})}_{\text{alto}} - \underbrace{(y_L^{\text{T,F}} - y_L^{\text{N,F}})}_{\text{bajo}}]}_{>0} \tag{1.119}$$

Nota que si el país doméstico es más productivo, y por lo tanto más rico, que el país extranjero, es probable que esta ventaja productiva se concentre en la producción de los bienes comercializables. Por esta razón, es razonable suponer que $y_L^{\text{T,H}} - y_L^{\text{N,H}}$ esté más alto que $y_L^{\text{T,F}} - y_L^{\text{N,F}}$, lo cual implica que el término entre los corchetes de la ecuación (1.119) es positivo (como indicado).

1.10.2 La hipótesis de Balassa-Samuelson

La hipótesis Balassa-Samuelson es una teoría que intenta explicar el efecto Balassa-Samuelson, es decir, la observación empírica de que los precios tienden a ser más altos en países ricos y más bajos en países pobres.

La hipótesis Balassa-Samuelson se basa en el supuesto que los precios de bienes comercializables, una vez convertidos a la misma moneda, no pueden ser diferentes entre países. Entonces el tipo de cambio real de los bienes comercializables es igual a cero:

$$q^{\text{T}} = s + p^{\text{T,H}} - p^{\text{T,F}} = 0. \tag{1.120}$$

Nota que esto implica que el tipo de cambio nominal tiene que ser igual a la ratio de los poderes adquisitivos de las monedas doméstica y extranjera:

$$s = -(p^{\text{T,H}} - p^{\text{T,F}}). \tag{1.121}$$

En este caso, el tipo de cambio real refleja la diferencia entre las productividades relativas de los sectores de los bienes comercializables de ambos países:

$$\begin{aligned} q &= s + p^{\text{H}} - p^{\text{F}} &(1.122)\\ &= s + p^{\text{T,H}} - p^{\text{T,F}} + \alpha[(p^{\text{N,H}} - p^{\text{N,F}}) - (p^{\text{T,H}} - p^{\text{T,F}})] &(1.123)\\ &= \alpha[(y_L^{\text{T,H}} - y_L^{\text{N,H}}) - (y_L^{\text{T,F}} - y_L^{\text{N,F}})] &(1.124)\\ &> 0. &(1.125) \end{aligned}$$

1.10.3 El modelo de flujos de divisas

La presión en el mercado de divisas versus la hipótesis de Balassa-Samuelson

Supongamos ahora que el tipo de cambio nominal, en lugar de estar determinado por la ecuación (1.121), se determine como en el modelo de flujos de divisas:

$$s = -(p^{\text{T,H}} - p^{\text{T,F}}) + \xi m^{\text{HF}}. \tag{1.126}$$

En este caso, el tipo de cambio real de los bienes comercializables ya no tiene que ser igual a cero:

$$\begin{aligned} q^{\text{T}} &= s + p^{\text{T,H}} - p^{\text{T,F}} &(1.127)\\ &= -(p^{\text{T,H}} - p^{\text{T,F}}) + \xi m^{\text{HF}} + p^{\text{T,H}} - p^{\text{T,F}} &(1.128)\\ &= \xi m^{\text{HF}} &(1.129)\\ &= \text{CMP}, &(1.130) \end{aligned}$$

donde CMP representa la presión en el mercado de divisas. En otras palabras, los precios de los bienes comercializables pueden diferir entre países.

El tipo de cambio real ahora viene dado por la media ponderada de la presión en el mercado de divisas por un lado y la diferencia entre las productividades relativas en los sectores de los bienes comercializables por el otro lado:

$$\begin{aligned} q &= s + p^{\text{H}} - p^{\text{F}} &(1.131)\\ &= s + p^{\text{T,H}} - p^{\text{T,F}} + \alpha[(p^{\text{N,H}} - p^{\text{N,F}}) - (p^{\text{T,H}} - p^{\text{T,F}})] &(1.132) \end{aligned}$$

$$= q^{\mathrm{T}} \qquad + \alpha[(y_L^{\mathrm{T,H}} - y_L^{\mathrm{N,H}}) - (y_L^{\mathrm{T,F}} - y_L^{\mathrm{N,F}})] \tag{1.133}$$
$$= \xi m^{\mathrm{HF}} \qquad + \alpha[(y_L^{\mathrm{T,H}} - y_L^{\mathrm{N,H}}) - (y_L^{\mathrm{T,F}} - y_L^{\mathrm{N,F}})]. \tag{1.134}$$

¿Y qué pasa si las productividades en los sectores de bienes comercializables y no comercializables no importan?

Nota que al determinar el tipo de cambio nominal, la ecuación (1.126) expresaba los poderes adquisitivos de las monedas doméstica y extranjera en términos de los precios de los bienes comerciables en ambos países (y no de los niveles generales de precios).

No obstante, si medimos los poderes adquisitivos de las monedas doméstica y extranjera en términos de los niveles generales de precios, vemos que el tipo de cambio real depende exclusivamente de la presión en el mercado de divisas:

$$s = -(p^{\mathrm{H}} - p^{\mathrm{F}}) + \xi m^{\mathrm{HF}}, \tag{1.135}$$
$$q = s + p^{\mathrm{H}} - p^{\mathrm{F}} \tag{1.136}$$
$$= -(p^{\mathrm{H}} - p^{\mathrm{F}}) + \xi m^{\mathrm{HF}} + p^{\mathrm{H}} - p^{\mathrm{F}} \tag{1.137}$$
$$= \xi m^{\mathrm{HF}}. \tag{1.138}$$

1.10.4 Precios en los sectores de bienes comercializables y no comercializables

Nota que para derivar el tipo de cambio real en el modelo de Balassa-Samuelson, no nos hacía falta calcular los niveles de precios en los sectores de bienes comercializables y no comercializables. Todo lo que teníamos que saber era la diferencia entre los precios de los bienes comercializables y no comercializables, que está inversamente relacionada con la diferencia entre las productividades en ambos sectores.

Aunque no nos hace falta saber los niveles de precios de los bienes comercializables y no comercializables, es posible calcularlos. Miremos el país doméstico, por ejemplo. Aquí tenemos tres ecuaciones: una para el nivel de precios de los bienes comercializables, $p^{\mathrm{T,H}}$, una para el nivel de precios de los bienes no comercializables, $p^{\mathrm{N,H}}$, y una para el nivel de precios general, p^{H}, que es la media (geométrica) ponderada de los niveles de precios en ambos sectores:

$$p^{\mathrm{T,H}} = w^{\mathrm{H}} - y_L^{\mathrm{T,H}}, \tag{1.139}$$
$$p^{\mathrm{N,H}} = w^{\mathrm{H}} - y_L^{\mathrm{N,H}}, \tag{1.140}$$
$$p^{\mathrm{H}} \;\; = (1-\alpha) p^{\mathrm{T,H}} + \alpha p^{\mathrm{N,H}}. \tag{1.141}$$

Nosotros suponemos que las productividades en los sectores de bienes comercializables y no comercializables vienen dadas de forma exógena por la tecnología que se emplea en ambos sectores. Además, podemos considerar el nivel de precios general como exógeno también, ya que el banco central tiene la posibilidad de controlarlo a través de su política monetaria. Entonces, las ecuaciones (1.139) a (1.141) representan tres ecuaciones, que se pueden utilizar para determinar las tres variables endógenas: el salario nominal, w^{H}, el nivel de precios de los bienes comerciables, $p^{\mathrm{T,H}}$, y el nivel de precios de los bienes no comerciables, $p^{\mathrm{N,H}}$.

El salario nominal, w^{H}, se puede determinar de la siguiente manera:

$$
\begin{aligned}
p^{\mathrm{H}} &= (1-\alpha)p^{\mathrm{T,H}} + \alpha p^{\mathrm{N,H}} && (1.142)\\
&= (1-\alpha)(w^{\mathrm{H}} - y_L^{\mathrm{T,H}}) + \alpha(w^{\mathrm{H}} - y_L^{\mathrm{N,H}}) && (1.143)\\
&= w^{\mathrm{H}} - (1-\alpha)y_L^{\mathrm{T,H}} - \alpha y_L^{\mathrm{N,H}}, && (1.144)\\
\Leftrightarrow \quad w^{\mathrm{H}} &= p^{\mathrm{H}} + (1-\alpha)y_L^{\mathrm{T,H}} + \alpha y_L^{\mathrm{N,H}}. && (1.145)
\end{aligned}
$$

Vemos que el salario real en el país doméstico, $w^{\mathrm{H}} - p^{\mathrm{H}}$, es la media ponderada de las productividades marginales de ambos sectores.

Además, se pueden determinar los niveles de precios de los bienes comercializables y no comercializables, $p^{\mathrm{T,H}}$ y $p^{\mathrm{N,H}}$, de la siguiente manera:

$$
\begin{aligned}
p^{\mathrm{T,H}} &= w^{\mathrm{H}} - y_L^{\mathrm{T,H}}, && (1.146)\\
&= p^{\mathrm{H}} - \alpha(y_L^{\mathrm{T,H}} - y_L^{\mathrm{N,H}}), && (1.147)\\
p^{\mathrm{N,H}} &= w^{\mathrm{H}} - y_L^{\mathrm{N,H}}, && (1.148)\\
&= p^{\mathrm{H}} + (1-\alpha)(y_L^{\mathrm{T,H}} - y_L^{\mathrm{N,H}}). && (1.149)
\end{aligned}
$$

Por lo tanto, descubrimos que cuando más alto es la productividad relativa del sector de los bienes comercializables, más bajos son los precios de los bienes comercializables y más altos son los precios de los bienes no comercializables en comparación con el nivel de precios general que fija el banco central.

1.11 Optimization

1.11.1 Optimization with equality constraints

$$\max f(x_1, x_2), \tag{1.150}$$

subject to

$$g(x_1, x_2) = b. \tag{1.151}$$

$$\max f(x_1, x_2, g(x_1, x_2)), \tag{1.152}$$

subject to

$$g(x_1, x_2) = b. \tag{1.153}$$

First-order conditions:

$$\begin{aligned} \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_1} &= f_1'(x_1, x_2, g(x_1, x_2)) + f_3'(x_1, x_2, g(x_1, x_2))g_1'(x_1, x_2) = 0, \\ \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_2} &= f_2'(x_1, x_2, g(x_1, x_2)) + f_3'(x_1, x_2, g(x_1, x_2))g_2'(x_1, x_2) = 0, \end{aligned} \tag{1.154}$$

where the chain rule is used.
Set $\lambda = f_3'(x_1, x_2, b)$, where the derivative is evaluated with x_1 and x_2 taking on the optimal values. Then, since $g(x_1, x_2) = b$, we can rewrite the first-order conditions as:

$$\begin{aligned} \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_1} &= f_1'(x_1, x_2) + \lambda g_1'(x_1, x_2) = 0, \\ \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_2} &= f_2'(x_1, x_2) + \lambda g_2'(x_1, x_2) = 0, \end{aligned} \tag{1.155}$$

where the functions f_1 and f_2 are written again with two arguments for simplicity.
These conditions have to be solved together with the constraint:

$$g(x_1, x_2) = b. \tag{1.156}$$

This gives three equations with three unknowns (x_1, x_2 and λ).

1.11.2 Optimization with inequality constraints

Problem

$$\max f(x_1, x_2), \tag{1.157}$$

subject to

$$g(x_1, x_2) \leq b. \tag{1.158}$$

Let's include the function g explicitly in the function f:

$$\max f(x_1, x_2, g(x_1, x_2)), \tag{1.159}$$

subject to

$$g(x_1, x_2) \leq b. \tag{1.160}$$

First-order conditions:

$$\begin{aligned} \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_1} &= f_1'(x_1, x_2, g(x_1, x_2)) + f_3'(x_1, x_2, g(x_1, x_2))g_1'(x_1, x_2) = 0, \\ \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_2} &= f_2'(x_1, x_2, g(x_1, x_2)) + f_3'(x_1, x_2, g(x_1, x_2))g_2'(x_1, x_2) = 0, \end{aligned} \tag{1.161}$$

where the chain rule is used.
Set $\lambda = f_3'(x_1, x_2, b)$, where the derivative is evaluated with x_1 and x_2 taking on the optimal values.
Note that it is necessary that $\lambda = f_3'(x_1, x_2, b) \geq 0$ (interior or corner solution).
Let's consider the two possible cases:

- In the case of a corner solution, the **constraint is "binding"** ($g(x_1, x_2) = b$) and $\lambda \geq 0$.

  Hence a solution has to satisfy:

$$f_1'(x_1, x_2, g(x_1, x_2)) + f_3'(x_1, x_2, g(x_1, x_2))g_1'(x_1, x_2) = 0, \tag{1.162}$$
$$f_2'(x_1, x_2, g(x_1, x_2)) + f_3'(x_1, x_2, g(x_1, x_2))g_2'(x_1, x_2) = 0, \tag{1.163}$$
$$g(x_1, x_2) = b, \tag{1.164}$$
$$\lambda \geq 0. \tag{1.165}$$

- In the case of an interior solution, the **constraint is not "binding"** ($g(x_1, x_2) < b$) and $\lambda = 0$. This means that the optimum must be the same no matter whether we optimize the functions f_1 and f_2 with two or with three arguments.

  Hence a solution has to satisfy:

$$f_1'(x_1, x_2) = 0, \tag{1.166}$$
$$f_2'(x_1, x_2) = 0, \tag{1.167}$$
$$g(x_1, x_2) < b, \tag{1.168}$$
$$\lambda = 0. \tag{1.169}$$

Hence we can find a solution by carrying out the following steps:

1. Set up the Lagrangian:

$$\mathcal{L}(x_1, x_2) = f(x_1, x_2) - \lambda g(x_1, x_2). \tag{1.170}$$

2. Check for possible solutions (that is, values for x_1, x_2 and λ), which have to satisfy:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 0, \tag{1.171}$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 0, \tag{1.172}$$
$$g(x_1, x_2) \leq b, \tag{1.173}$$
$$\lambda \geq 0 \quad (\lambda = 0 \quad \text{if} \quad g(x_1, x_2) < b). \tag{1.174}$$

3. Among the solution candidates, pick the one that maximizes the objective function.

Lagrange method

$$\max f(x_1, \ldots, x_n), \tag{1.175}$$

subject to

$$\begin{aligned} g_1(x_1, \ldots, x_n) &= b_1, \\ &\cdots \\ g_m(x_1, \ldots, x_n) &= b_m \end{aligned} \tag{1.176}$$

We assume that $m < n$.

By setting $\mathbf{x} = (x_1, \ldots, x_n)'$, the problem can be written more compactly as follows:

$$\max f(\mathbf{x}), \tag{1.177}$$

subject to

$$g_j(\mathbf{x}) = b_1, \quad j = 1, \ldots, m. \tag{1.178}$$

$$\tag{1.179}$$

To solve this problem, we set up the Lagrange function, or Lagrangian:

$$\mathcal{L}(\mathbf{x}) = f(\mathbf{x}) - \lambda_1 g_1(\mathbf{x}) - \ldots - \lambda_m g_m(\mathbf{x}), \tag{1.180}$$

where $\lambda_1, \ldots, \lambda_m$ are called Lagrange multipliers.
The first-order conditions are then:

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial x_i} = \frac{\partial f(\mathbf{x})}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\mathbf{x})}{\partial x_i} = 0, \quad i = 1, \ldots, n. \tag{1.181}$$

We can now find the solution to the optimization problem in equations (1.175) and (1.176) by solving simultaneously the n first-order conditions in equation (1.181) and the m constraints in equation set (1.176) for the $n + m$ variables $x_1, \ldots, x_n$ and $\lambda_1, \ldots, \lambda_m$.
Indeed, it can be shown that if we can find Lagrange multipliers $\lambda_1, \ldots, \lambda_m$ and a vector $\mathbf{x}^*$ that satisfy all of the first-order conditions and constraints and if the Lagrangian $\mathcal{L}(\mathbf{x})$ is concave in $\mathbf{x}$, then $\mathbf{x}^*$ solves the optimization problem in equations (1.175) and (1.176) (see, for example, Sydsæter, Hammond, Seierstad and Strøm, 2005).

Examples

Example 1 Consider the following optimization problem:

$$\max_x -x^2 + 2, \tag{1.182}$$

subject to

$$x = b. \tag{1.183}$$

The Lagrangian is:

$$\mathcal{L}(x) = -x^2 + 2 - \lambda x = 0. \tag{1.184}$$

The first-order condition is:

$$\frac{d\mathcal{L}(x)}{dx} = -2x - \lambda = 0. \tag{1.185}$$

Hence:

$$x^* = b, \tag{1.186}$$

$$\lambda = -2b. \tag{1.187}$$

Note that the Lagrange multiplier $\lambda = \lambda(b)$ is the rate at which the optimal value of the objective function changes with respect to changes in the constant b:

$$\lambda(b) = \frac{df^*(b)}{db} = \frac{df(x^*(b))}{db} = \frac{-(x^*)^2 + 2}{dx^*} \times \frac{dx^*(b)}{db} = -2x^*. \tag{1.188}$$

In this simple example, x^* always equals b. Hence $f^*(b) = -b^2 + 2$ and $\lambda(b) = -2b$. For example, when $b = -1$, $f^*(b)$ rises at rate 2 with respect to b. When $b = 0$, $f^*(b) = -b^2 + 2$ stays constant when b changes marginally. And when $b = 1$, $f^*(b)$ falls at rate -2 with respect to b.

To simplify the notation, from now on we will omit the asterisk for solutions (for example, x instead of x^*).

Example 2 Things become more interesting when there are two or more variables. Consider the following optimization problem with two variables:

$$\max_{C_1, C_2} \ln(C_1) + \ln(C_2), \tag{1.189}$$

subject to

$$C_1 + C_2 = Y. \tag{1.190}$$

Note that it is possible to transform this problem into a problem of just one variable:

$$\max_{C_1, C_2} \ln(C_1) + \ln(Y - C_1). \tag{1.191}$$

The first-order condition is:

$$\frac{1}{C_1} - \frac{1}{Y - C_1} = 0. \tag{1.192}$$

Hence the solution is:

$$C_1 = C_2 = \frac{1}{2}Y. \tag{1.193}$$

The Lagrangian yields the same result, but provides us also with the shadow value of the constraint. The Lagrangian is:

$$\mathcal{L}(C_1, C_2) = \ln(C_1) + \ln(C_2) - \lambda(C_1 + C_2). \tag{1.194}$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}(C_1, C_2)}{\partial C_1} = \frac{1}{C_1} - \lambda = 0, \tag{1.195}$$

$$\frac{\partial \mathcal{L}(C_1, C_2)}{\partial C_2} = \frac{1}{C_2} - \lambda = 0. \tag{1.196}$$

Again the solution is:

$$C_1 = C_2 = \frac{1}{2}Y, \tag{1.197}$$

but now we also find out that the marginal benefit of relaxing the constraint is:

$$\lambda = \frac{2}{Y}. \tag{1.198}$$

Bibliography

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Sydsæter, Knut, Peter Hammond, Atle Seierstad and Arne Strøm. *Further Mathematics for Economic Analysis*. Pearson Education, Essex, 2005.